AN ARTIN–MUMFORD CRITERION FOR CONIC BUNDLES IN CHARACTERISTIC TWO

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ABSTRACT. We prove a characteristic two version of the famous criterion of Artin and Mumford for irrationality of conic bundles. On the one hand, combined with the pathological behaviour of conic bundles in characteristic two, this allows us to construct easier and more explicit examples of irrational conic bundles. On the other hand, degeneration techniques *à la Voisin* allow to deduce irrationality results in characteristic zero.

CONTENTS

1. INTRODUCTION

The Artin–Mumford criterion for irrationality of conic bundles. One of the fundamental problem in algebraic geometry is to understand whether a variety X over an algebraically closed field k is stably rational, i.e. whether $X \times \mathbb{P}^n$ is birational to $\mathbb{P}^m$ for some $m, n \in \mathbb{N}$. One of the first and most influential stable irrationality criterion, has been proved by Artin and Mumford. Since it is the starting point of this paper, we recall it now.

Theorem 1.1. [AM72] *Let k be an algebraically closed field of characteristic different from 2. Let $f : X \to B$ be a flat conic bundle over k between smooth, proper and connected k-varieties. Assume the following.*

(1) The discriminant is disconnected.
(2) There are two distinct connected components of the discriminant on which all the fibers are reduced and such that the conic bundle over each of them is not a product.
(3) The group $H^3_{ét}(B, \mathbb{Z}_2)$ vanishes.

Then X is not stably rational.

Since then, many other techniques have been developed to study rationality problems and we focus here on two main approaches.

On the one hand, Theorem 1.1 has been extended via the introduction of unramified cohomology (with $\mathbb{Z}/2$-coefficients) by Colliot-Thélène and Ojanguren [CTO89]. This has been further developed and combined with degenerations techniques *à la Voisin* [Voi15, CTP16], leading to striking applications (culminating with the work of Schreieder [Sch19]). These techniques proved to be very powerful, although constructing examples with them becomes tricky. Also, these results do not work in characteristic 2, since 2-torsion étale cohomology is quite badly behaved.

On the other hand, techniques using reduction modulo a prime number p, especially with $p = 2$, have been developed by Kollar [Kol95] and used by Totaro [Tot16], by exploiting pathological behaviour of differential forms in positive characteristic. These techniques have the advantage of having a more geometric flavour and of being more elementary in nature.

The Artin–Mumford criterion in characteristic 2. In this paper, we take a first step in trying to combine both approaches, by extending Theorem 1.1 to characteristic 2 and showing that this extension is often easier to apply to concrete examples. Moreover, using degeneration techniques *à la Voisin*, we deduce irrationality results in characteristic zero; see for instance Theorem 1.4. Our main result is the following.

Theorem 1.2. *Let k be an algebraically closed field of characteristic 2. Let $f : X \to B$ be a flat and dominant conic bundle over k, with smooth generic fiber, between smooth, proper and connected k-varieties. Assume the following.*

(1) The discriminant is disconnected.
(2) There are two disctinct connected components of the discriminant of f, each of which contains a fiber which is not reduced.
(3) The group $\mathrm{H}^2(B, \Omega^1_{B/k})$ vanishes.

Then X is not stably rational.

Comparison between the two Artin–Mumford criteria. Let us enlighten the main differences between Theorem 1.1 and Theorem 1.2. First of all, in our result, the field k is of characteristic 2, so it covers the case left open by Theorem 1.1. Secondly, and most importantly, the assumptions (2) are different. In practice, checking assumption (2) in Theorem 1.1 is quite complicated since one has to understand the behaviour of the fibers in a family, while assumption (2) in Theorem 1.2 is immediate to check in concrete examples. There is also a version of Theorem 1.2 with reduced fibers over the discriminant (see Theorem 5.2), which is more similar to the original statement of Artin and Mumford, but one shall consider Theorem 1.2 as more useful in practice, because no hypothesis on the nontriviality of the family appears.

One could also state and prove Theorem 1.2 in other characteristics, but constructing nonreduced fibers there is much harder. Indeed, locally in B, the conic bundle has equation

$$\alpha a^2 + \beta b^2 + \gamma c^2 + \alpha' bc + \beta' ac + \gamma' ab = 0$$

for some local functions $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ on B. In characteristic two, the locus of nonreduced fiber is simply given by the equations $\alpha' = \beta' = \gamma' = 0$, while in characteristic different from 2 it has a complicated expression (see [Tan24, Definition 3.6]).

The invariant used by Artin and Mumford to obstruct rationality is the 2-torsion in étale cohomology. It turns out that the nonzero class they construct is algebraic. In our case we follow their approach and construct a similar 2-torsion algebraic class. The only cohomology in characteristic two which might have nonzero 2-torsion is crystalline cohomology, hence we have to work there. This adds various technical complications, since one has to deal with differential forms on conic bundles in characteristic 2. To do this, we will need to work with complexes and to control the difference between relative and asbolute diffrential forms (see Proposition 4.4).

Applications. We construct explicit examples to which Theorem 1.2 applies. Given the special shape of the discriminant in characteristic 2, this can be achieved quite easily.

Theorem 1.3. *Let k be a field of characteristic two. Let $B = \mathbb{P}^2$ with coordinates x, y, z. Consider the conic bundle $f : X \to B$ defined in $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(3)$ by*

$$a^2 + xab + yzb^2 + (x(y^3 + z^3) + y^2z^2)bc + (y^6 + z^6 + x^4yz + xz^5 + xy^5)c^2 = 0.$$

Then X is not stably rational.

From Theorem 1.3 and the other examples we construct, we deduce the following.

Theorem 1.4. *Let k be a field of characteristic zero or two. A very general conic bundle in $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(3))$ over $\mathbb{P}^2_k$ with values in $\mathcal{O}$ has no decomposition of the diagonal, hence it is not stably rational.*

Similarly, a very general conic bundle in $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(3))$ over $\mathbb{P}^2_k$ with values in $\mathcal{O}(1)$ has no decomposition of the diagonal, hence it is not stably rational.

As far as we are aware of, this result is new in characteristic two, but it seems likely that it can be proved using other obstructions in characteristic zero. We believe that much more interesting examples can be constructed in characteristic two, hence giving new results in characteristic zero, see for instance Remark 8.9.

Remark 1.5. A different irrationality criterion for conic bundle in characteristic 2 has been proved in [ABBGvB21]. Compared to the geometric Theorem 1.2, their criterion is closer to the unramified cohomological interpretation of Theorem 1.1 given in [CTO89]. Since unramified cohomology with 2-torsion coefficients does not work well in characteristic 2, their criterion is quite involved to check in practice and it needs the explicit construction of an auxiliary conic bundle. In Example 8.8, we show how to interpret their main concrete application via Theorem 1.2.

Organization of the paper. Section 2 contains some conventions. Section 3 recalls the definition of conic bundle and the one of its discriminant. This is subtle in characteristic two. Section 4 is a technical section doing computations on the cohomology of sheaves of differential forms on conic bundles. These computations are used in Section 5 where the crystalline version of the Artin–Mumford criterion is stated and proved. Section 6 explains a birational transformations which makes the disconnectedness hypothesis (1) of Theorem 1.2 easier to be checked. In Section 7 we recall the notion of the decomposition of the diagonal, its relation to rationality problems and explain why torsion in crystalline cohomology obstructs the decomposition of the diagonal. Putting this discussion together with the crystalline Artin–Mumford criterion, we give geometric conditions that obstruct rationality. Finally, in Section 8 we give explicit examples where these geometric conditions are satisfied and deduce irrationality results for some very general varieties in characteristic zero and two. We also discuss there examples one might hope to construct in the future.

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2. Notation and convention

Throughout the paper we use the following conventions.

(1) A variety is a separated scheme of finite type over a field. When the field k is specified we call this a k-variety.
(2) A conic over a field k is a subscheme of $\mathbb{P}^2_k$ defined by a global section of $\mathcal{O}(2)$.
(3) A conic which is geometrically isomorphic to $xy = 0$ in $\mathbb{P}^2$, is a cross. A conic which is geometrically isomorphic to $x^2 = 0$ in $\mathbb{P}^2$, will be called a double line. See also Definition 3.4.
(4) If k is a field and $\mathcal{F}$ is a coherent sheaf on a proper k-scheme S, we write $h^i_S(\mathcal{F})$ for the dimension of the k-vector space $\mathrm{H}^i(S, \mathcal{F})$ and we drop the index S if it is clear from the context.
(5) If X is a k-variety and $\mathcal{E}$ is vector bundle of rank r on X, we write $\mathbb{P}(\mathcal{E}) \to X$ for the associated projective bundle with fibers $\mathbb{P}^{r-1}$.
(6) The Chow groups $\mathrm{CH}(X)$ of algebraic cycles modulo rational equivalence of a variety X has always integral coefficients.
(7) For an object X (a variety, a sheaf,...) living on some base B, we write X_S for the base change of X to S when the map from S to B is clear.
(8) For a prime number p and an abelian group M, the p-torsion of M is denoted by $M[p]$.
(9) If X is a variety over a field k of positive characteristic, $\mathrm{H}^i_{\mathrm{crys}}(X) := \mathrm{H}^i_{\mathrm{crys}}(X, W(k))$ denotes the crystalline cohomology with integral coefficients.
(10) The projective plane will sometimes be the base and sometimes be a fiber of a fibration. With the hope of clearly distinguishing the two situations, we use a, b, c as variables on the fiber and x, y, z as variables on the base.

3. Discriminant of conic bundles in characteristic two

This section gives generalities on conic bundles with particular emphasis to characteristic two. This is all taken from [ABBGvB21] and [Tan24] and recalled here for the convenience of the reader. We fix an algebraically closed field k and a connected smooth k-variety B.

Quadratic forms on rank 3 vector bundles. Let $\mathcal{E}$ be a rank 3-vector bundle on B. Let $S_2(\mathcal{E}) \subset \mathcal{E} \otimes \mathcal{E}$ be the set of symmetric tensors and $\mathcal{E} \otimes \mathcal{E} \twoheadrightarrow S^2(\mathcal{E})$ the symmetric power of $\mathcal{E}$. By construction, there are identifications $S_2(\mathcal{E})^\vee \simeq S^2(\mathcal{E}^\vee)$.

Definition 3.1. Let $\mathcal{L}$ be a line bundle on B. A quadratic form on $\mathcal{E}$ with values in $\mathcal{L}$ is a morphism $q : S_2(\mathcal{E}) \to \mathcal{L}$.

By duality, giving a quadratic form on $\mathcal{E}$ with values in $\mathcal{L}$ is equivalent to give a global section of $S^2(\mathcal{E}^\vee) \otimes \mathcal{L}$.

Remark 3.2. Classically, quadratic forms are maps $q : \mathcal{E} \to \mathcal{L}$ satisfying $q(av) = a^2 v$ for local sections a of $\mathcal{O}_B$ and v of $\mathcal{E}$ and such that the map $\varphi : \mathcal{E} \times \mathcal{E} \to \mathcal{L}$ defined on local sections by $\varphi(x,y) := q(x+y) - q(y) - q(x)$ is bilinear. To recover this notion from the one defined before, one observes that the natural map sending $q : S_2(\mathcal{E}) \to \mathcal{L}$ to the map $\widetilde{q} : E \to \mathcal{L}$ defined on local section by $\widetilde{q}(v) = q(v \otimes v)$, gives a bijection between classical quadratic forms and the one we defined (see e.g. [Woo09, Proposition 2.6.1] or [Aue11, Lemma 1.1]).

Conic bundles. Let $q : S_2(\mathcal{E}) \to \mathcal{L}$ be a quadratic form on a vector bundle $\mathcal{E}$ on B with values in a line bundle $\mathcal{L}$ on B and let $\pi : \mathbb{P}(\mathcal{E}) \to B$ be the canonical map. Since

$$\mathrm{H}^0(B, S^2(\mathcal{E}^\vee) \otimes \mathcal{L}) \simeq \mathrm{H}^0(B, \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{L})) \simeq \mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^* L),$$

every nonzero quadratic form $q : S_2(\mathcal{E}) \to \mathcal{L}$ defines a closed subscheme $X_q \subset \mathbb{P}(\mathcal{E})$ whose natural morphism $f_q : X_q \to B$ has generic fiber a conic.

Definition 3.3. Let $f : X \to B$ be a morphism between k-varieties. We say that f is a conic bundle if there exists a rank 3 vector bundle $\mathcal{E}$ on B, a line bundle $\mathcal{L}$ on B and a quadratic form $q : S_2(\mathcal{E}) \to \mathcal{L}$ such that $f : X \to B$ is isomorphic to $f_q : X_q \to B$. In this case, we say that X is a conic bundle in $\mathbb{P}(\mathcal{E})$ over B with values in $\mathcal{L}$.

Discriminant and locus of double lines. Let a, b, c be the coordinates in $\mathbb{P}^2$. Over an algebraically closed field, up to a change of coordinates, a conic C is defined by one of the following equations (see e.g. [ABBGvB21, Corollary 2.5]):

$$(1) \quad a^2 + bc; \qquad (2) \quad ab; \qquad (3) \quad a^2.$$

In case (1), C is smooth. In case (2), C it is the wedge of two $\mathbb{P}^1$. In case (3), C is irreducible but not reduced.

Definition 3.4. With the above notations, a conic of type (2) will be called a cross and a conic of type (3) will be called a double line.

Let $f : X \to B$ be a flat conic bundle. We will denote by Δ the discriminant of f (the locus of singular fibers) and by Σ the locus of double lines of f.

Remark 3.5. In [Tan24, Definition 3.4-3.10], Tanaka defined closed subschemes $\Sigma \subseteq \Delta \subseteq B$ such that X_b is singular if and only if $b \in \Delta$ (i.e. the geometric fiber is of type (2) or (3)) and the geometric fiber is moreover of type (3) if and only if $b \in \Sigma$. In general, both Δ and Σ might be nonreduced or might be equal to B. If $X \to B$ is generically smooth, then Δ is a Cartier divisor in B. For our purposes, we will only need the reduced structure of Δ and Σ.

Direct sum of line bundles. Assume that $\mathcal{E} \simeq \mathcal{E}_a \oplus \mathcal{E}_b \oplus \mathcal{E}_c$ is the direct sum of three line bundles $\mathcal{E}_i$ and write $\mathcal{E}_{i,j} := \mathcal{E}_i \otimes \mathcal{E}_j$. In this case we have

$$S_2(\mathcal{E}) \simeq \mathcal{E}_{a,a} \oplus \mathcal{E}_{b,b} \oplus \mathcal{E}_{c,c} \oplus \mathcal{E}_{a,b} \oplus \mathcal{E}_{a,c} \oplus \mathcal{E}_{b,c}$$

and

$$S_2(\mathcal{E}^\vee) \simeq \mathcal{E}^\vee_{a,a} \oplus \mathcal{E}^\vee_{b,b} \oplus \mathcal{E}^\vee_{c,c} \oplus \mathcal{E}^\vee_{a,b} \oplus \mathcal{E}^\vee_{a,c} \oplus \mathcal{E}^\vee_{b,c}$$

Hence to give a quadratic form on $\mathcal{E}$ with values in $\mathcal{L}$, is equivalent to give a collection of six global sections $s_{i,j} \in \mathrm{H}^0(B, \mathcal{E}^\vee_{i,j} \otimes \mathcal{L})$ for $i \leq j$. We represent this situation with a "half" matrix:

	$\mathcal{E}_a^\vee$	$\mathcal{E}_b^\vee$	$\mathcal{E}_c^\vee$
$\mathcal{E}_a^\vee$	$s_{a,a}$		
$\mathcal{E}_b^\vee$	$s_{a,b}$	$s_{b,b}$	
$\mathcal{E}_c^\vee$	$s_{a,c}$	$s_{b,c}$	$s_{c,c}$

corresponding to the conic bundle of equation

$$s_{a,a}a^2 + s_{b,b}b^2 + s_{c,c}c^2 + s_{a,b}ab + s_{a,c}ac + s_{b,c}bc = 0.$$

In this case the discriminant divisor is given by the zero locus of

$$\Delta = 4s_{a,a}s_{b,b}s_{c,c} + s_{a,b}s_{b,c}s_{a,c} - s_{a,b}^2 s_{c,c} - s_{a,c}^2 s_{b,b} - s_{b,c}^2 s_{a,a} = 0.$$

The locus of double lines Σ is much more complicated in general. See [Tan24, Proposition 3.12] for an explicit formula.

Remark 3.6. When k is of characteristic 2, Δ simplifies to

$$s_{a,b}s_{b,c}s_{a,c} + s_{a,b}^2 s_{c,c} + s_{a,c}^2 s_{b,b} + s_{b,c}^2 s_{a,a} = 0.$$

Moreover, the locus Σ of double lines is simply given by the equation

$$s_{a,b} = s_{a,c} = s_{b,c} = 0.$$

These simplified formulas are the reason for which it is easier to construct examples where Theorem 1.2 applies than where the original Artin–Mumford criterion applies.

4. Cohomology of conic bundles

In this section we collect preliminaries on the cohomology of conics and conic bundles. We will fix an algebraically closed field k and a connected smooth k-variety B.

Lemma 4.1. *Let $i : C \hookrightarrow \mathbb{P}^2_k$ be a conic defined by a sheaf of ideal $\mathcal{I}_C \simeq \mathcal{O}(-2)$. Then the following computations hold.*

(1) $h^0_C(\mathcal{O}_C) = 1$ and $h^1_C(\mathcal{O}_C) = 0$.
(2) $h^0_C(i^*\Omega^1_{\mathbb{P}^2/k}) = 0$ and $h^1_C(i^*\Omega^1_{\mathbb{P}^2/k}) = 4$.
(3) $h^0_C(i^*\mathcal{I}_C) = 0$ and $h^1_C(i^*\mathcal{I}_C) = 3$.

Proof. Part (1) follows from the exact sequence of coherent sheaves on $\mathbb{P}^2$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0 \tag{4.2}$$

and the fact that

$$h^i_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0 \text{ for } i \geq 0, \quad h^0_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}) = 1, \quad h^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}) = h^2_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}) = 0.$$

Now recall the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^2/k} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

Pulling back to C, we find an exact sequence

$$0 \to i^*\Omega^1_{\mathbb{P}^2/k} \to i^*\mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_C \to 0$$

so that, thanks to (1), part (2) is reduced to show that

$$h^0_C(i^*\mathcal{O}_{\mathbb{P}^2}(-1)) = 0, \quad \text{and} \quad h^1_C(i^*\mathcal{O}_{\mathbb{P}^2}(-1)) = 1.$$

These in turn, follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-1) \to i_*\mathcal{O}_C(-1) \to 0,$$

obtained by tensoring (4.2) with $\mathcal{O}_{\mathbb{P}^2}(-1)$, and the fact that

$$h^i_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0 \text{ for } i \geq 0, \quad h^0_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-3)) = h^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-3)) = 0, \quad h^2_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-3)) = 1.$$

For (3), tensoring (4.2) with $\mathcal{I}_C \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-2) \to i_*i^*\mathcal{I}_C \to 0,$$

so the conclusion follows from the equalities

$$h^i_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0 \text{ for } i \geq 0, \quad h^0_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-4)) = h^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-4)) = 0, \quad h^2_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-4)) = 3.$$

□

We now turn to some computations on the cohomology conic bundles. We will make an extensive use of the following classical theorem; see [Mum08, Chapter II, §5, Corollary 2].

Theorem 4.3. *Let $f : X \to B$ be a flat proper morphism and let $\mathcal{F}$ be a coherent sheaf over X flat over B. If $h^i_{X_p}(\mathcal{F}_p)$ is constant for every geometric point $p \in B$, then $R^i f_*\mathcal{F}$ is locally free and the natural map*

$$R^i f_*\mathcal{F} \otimes k(p) \to \mathrm{H}^i(X_p, \mathcal{F}_p)$$

is an isomorphism.

Proposition 4.4. *Let X be a smooth k-variety and let $f : X \to B$ be a flat conic bundle with smooth generic fiber. Then*

(1) $f_*\mathcal{O}_X = \mathcal{O}_B$ and $R^i f_*\mathcal{O}_X = 0$ for $i > 0$;
(2) The right exact sequence

$$f^*\Omega^1_{B/k} \to \Omega^1_{X/k} \to \Omega^1_{X/B} \to 0.$$

is also left exact.
(3) $f_*\Omega^1_{X/B} = 0$
(4) The natural maps

$$\Omega^1_{B/k} \to f_*\Omega^1_{X/k} \quad \text{and} \quad R^1 f_*\Omega^1_{X/k} \to R^1 f_*\Omega^1_{X/B}$$

are isomorphism.

Proof. Part (1) follows directly from Theorem 4.3 and Lemma 4.1(1) (see also [Tan24, Lemma 2.5]).

For (2), we have to show that the map $f^*\Omega^1_{B/k} \to \Omega^1_{X/k}$ is injective. Since both sheaves are locally free, it is enough to check injectivity after the restriction to an open subset of X. As $f : X \to B$ is generically smooth, we can assume that f is smooth and in this case the sequence is also left exact (see e.g. [Sta18, Tag 02K4]).

Let us now show (3). Let $\mathcal{E}$ be a rank 3 vector bundle on B as in Definition 3.3. In particular, there is a closed immersion $i : X \to \mathbb{P}(\mathcal{E})$ over B, such that, for every geometric point $p \in B$, the inclusion $i_p : X_p \to \mathbb{P}(\mathcal{E})_p = \mathbb{P}^2$ is the anticanonical embedding of X_p. Let $\mathcal{I}_X$ be the sheaf ideal of X in $\mathbb{P}(\mathcal{E})$. We have a right exact sequence

$$i^*\mathcal{I}_X \to i^*\Omega^1_{\mathbb{P}(\mathcal{E})/B} \to \Omega^1_{X/B} \to 0$$

by [Sta18, Tag 01UZ]. We claim that it is also left exact, i.e. that $i^*\mathcal{I}_X \to i^*\Omega^1_{\mathbb{P}(\mathcal{E})/B}$ is injective. Indeed, since $X \subseteq \mathbb{P}(\mathcal{E})$ is a local complete intersection, $i^*\mathcal{I}_X$ is locally free ([Sta18, Tag 06B9]). Since also $\Omega^1_{\mathbb{P}(\mathcal{E})/B}$ is locally free, it is enough to show that $i^*\mathcal{I}_X \to i^*\Omega^1_{\mathbb{P}(\mathcal{E})/B}$ is injective on an open subset. Hence, we can assume that f is smooth, where the conclusion follows from [Sta18, Tag 06AA].

We can now pushforward via $f : X \to B$ the short exact sequence

$$0 \to i^*\mathcal{I}_X \to i^*\Omega^1_{\mathbb{P}(\mathcal{E})/B} \to \Omega^1_{X/B} \to 0$$

to find a long exact sequence

$$0 \to f_* i^* \mathcal{I}_X \to f_* i^* \Omega^1_{\mathbb{P}(\mathcal{E})/B} \to f_* \Omega^1_{X/B} \to R^1 f_* i^* \mathcal{I}_X \to R^1 f_* i^* \Omega^1_{\mathbb{P}(\mathcal{E})/B} \to R^1 f_* \Omega^1_{X/B} \to 0.$$

By Theorem 4.3 and Lemma 4.1(2), $f_* i^* \Omega^1_{\mathbb{P}(\mathcal{E})/B}$ vanishes. Hence $f_* \Omega^1_{X/B}$ identifies with the kernel of the map

$$R^1 f_* i^* \mathcal{I}_X \to R^1 f_* i^* \Omega^1_{\mathbb{P}(\mathcal{E})/B}.$$

We want to show that the map is injective. Since $X \to B$ is flat and $\mathcal{I}_X$ is flat over B, the restriction of $\mathcal{I}_X$ to $\mathbb{P}(\mathcal{E})_p$ identifies with the ideal defining X_p in $\mathbb{P}(\mathcal{E})_p$. Hence, by Lemma 4.1 and Theorem 4.3, the coherent sheaves $R^1 f_* i^* \mathcal{I}_X$ and $R^1 f_* i^* \Omega^1_{\mathbb{P}(\mathcal{E})/B}$ are locally free, so it is enough to show injectivity on an open, hence to show the vanishing of $f_* \Omega^1_{X/B}$ on an open. We can then assume that f is smooth, in which case the fibers of $f_* \Omega^1_{X/B}$ identifies with $\mathrm{H}^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1/k}) = 0$, so we conclude again by Theorem 4.3.

Finally, let us now show (4). By (2), we have a short exact sequence

$$0 \to f^* \Omega^1_{B/k} \to \Omega^1_{X/k} \to \Omega^1_{X/B} \to 0.$$

Pushing forward, we get a long exact sequence

$$0 \to f_* f^* \Omega^1_{B/k} \to f_* \Omega^1_{X/k} \to f_* \Omega^1_{X/B} \to R^1 f_* f^* \Omega^1_{B/k} \to R^1 f_* \Omega^1_{X/k} \to R^1 f_* \Omega^1_{X/B} \to 0.$$

By the projection formula and (1)

$$f_* f^* \Omega^1_{B/k} \simeq \Omega^1_{B/k} \otimes f_* \mathcal{O}_X \simeq \Omega^1_{B/k} \quad \text{and} \quad R^1 f_* f^* \Omega^1_{B/k} \simeq \Omega^1_{B/k} \otimes R^1 f_* \mathcal{O}_X = 0,$$

hence $R^1 f_* \Omega^1_{X/k} \simeq R^1 f_* \Omega^1_{X/B}$ and there is a short exact sequence

$$0 \to \Omega^1_{B/k} \to f_* \Omega^1_{X/k} \to f_* \Omega^1_{X/B} \to 0.$$

So the conclusion follows from (3). □

Corollary 4.5. *Keep notation from the above theorem. Then the following holds.*

(1) $h^i_B(\mathcal{O}_B) = h^i_X(\mathcal{O}_X)$ *for all* $i > 0$ *and* $h^0_B(\Omega^1_{B/k}) = h^0_X(\Omega^1_{X/k})$.

(2) The natural map $\mathrm{H}^0(B, \Omega^i_{B/k}) \to \mathrm{H}^0(X, \Omega^i_{X/k})$ *is injective for all* $i > 0$.

Proof. Part (1) follows from the Leray spectral sequence for $f : X \to B$ and Proposition 4.4(1-3-4).

Let us now show (2). By Proposition 4.4(2) the natural map

$$f^* \Omega^1_{B/k} \hookrightarrow \Omega^1_{X/k}.$$

is injective. Since these sheaves are locally free, we can pass to exterior powers and deduce that the natural map

$$f^* \Omega^i_{B/k} \hookrightarrow \Omega^i_{X/k}.$$

is injective. By the projection formula and Proposition 4.4(1), taking global sections gives the conclusion. □

Corollary 4.6. *Keep notation from the above theorem. Assume in addition that* $h^i_B(\mathcal{O}_B)$ *and* $h^0_B(\Omega^i_{B/k})$ *vanish for* $i > 0$. *Then the following holds.*

(1) $h^i_X(\mathcal{O}_X) = 0$ *for every* $i > 0$ *and* $h^0_X(\Omega^1_{X/k}) = 0$

(2) If one has also $h^0_X(\Omega^i_{X/k}) = 0$ *for* $i > 0$ *then* $\mathrm{H}^1(X, \Omega^1_{X/k}) = \mathrm{H}^2_{\mathrm{dR}}(X)$.

Proof. Part (1) follows from Corollary 4.5(1).

Let us now show (2). By (1) and the assumption $h^0_X(\Omega^i_{X/k}) = 0$ it is enough to show that $h^1_X(\Omega^1_{X/k}) = \dim_k(\mathrm{H}^2_{\mathrm{dR}}(X))$. Let $\mathcal{H}^i_X$ be the sheaf $\mathrm{H}^i(\Omega^\bullet_X)$ on X, so that there is the conjugate spectral sequence

$$E_2^{a,b} := \mathrm{H}^a(X, \mathcal{H}^b_X) \Rightarrow H^{a+b}_{\mathrm{dR}}(X).$$

The Cartier isomorphism [Kat70, Theorem 7.2] shows that

$$\dim_k(\mathrm{H}^a(X, \mathcal{H}^b_X)) = \dim_k(\mathrm{H}^a(X, \Omega^b_{X/k}))$$

In particular, by assumption, $E_2^{0,b}$ for $b > 0$, so that there are no non trivial morphism from or to $\mathrm{H}^1(X, \mathcal{H}^1_X)$, hence $E_\infty^{1,1} = \mathrm{H}^1(X, \mathcal{H}^1_X)$. Moreover, by hypothesis, $E_2^{2,0}$ vanishes as well, so we have

$$\mathrm{H}^1(X, \mathcal{H}^1_X) \simeq H^2_{\mathrm{dR}}(X).$$

To conclude the proof just observe that

$$\dim_k(\mathrm{H}^1(X, \mathcal{H}^1_X)) = h^1_X(\Omega^1_{X/k})$$

again by Cartier. □

5. Crystalline Artin–Mumford criterion in characteristic two

In this section we prove a characteristic 2 version of the Artin–Mumford theorem, stating that, under some hypothesis on the discriminant, the total space of a conic bundle has torsion in its cohomology. In Section 7 we will see that such a cohomological consequence is an obstruction to the decomposition of the diagonal, hence to stable rationality.

We will make use of the locus of crosses and double lines on the base of a conic bundles, as introduced in Definition 3.4.

Definition 5.1. Let $f : X \to B$ be a flat conic bundle with smooth generic fiber (Definition 3.3). Let D be a closed subvariety of the discriminant (Definition 3.4). We say that D is Artin–Mumford if one of the following conditions holds.

(1) There exists a point $p \in D$, such that the fiber X_p is a double line.
(2) All fibers above D are crosses, D is smooth and the fibration $X_D \to D$ is not a product.

Theorem 5.2. *Let k be an algebraically closed field of characteristic 2. Let $f : X \to B$ be a flat conic bundle over k with smooth generic fiber between smooth, proper and connected k-varieties. Assume the follwing.*

(1) The discriminant is disconnected.
(2) There are two disctinct connected components of the discriminant of f which are Artin–Mumford (Definition 5.1).
(3) The group $\mathrm{H}^2(B, \Omega^1_{B/k})$ vanishes.
(4) The groups $\mathrm{H}^i(B, \mathcal{O}_B)$ and $\mathrm{H}^0(B, \Omega^i_{B/k})$ vanish for $i > 0$.
(5) The groups $\mathrm{H}^0(X, \Omega^i_{X/k})$ vanish for $i > 0$.

Then $\mathrm{H}^{2d-2}_{\mathrm{crys}}(X)[2] \neq 0$, where d is the dimension of X.

Proof. Let $\alpha \in \mathrm{H}^{2d-2}_{\mathrm{crys}}(X)$ be the class from Definition 5.6. It is 2-torsion by Lemma 5.7. On the other hand it is nonzero by Lemma 5.8. □

Remark 5.3. The first three hypothesis in the above theorem are analogous to those of the classical Artin–Mumford criterion, see Theorem 1.1. Hypothesis (4) can probably be avoided with a more complicated proof but it is in practice verified by all the interesting examples (e.g. B rational). Hypothesis (5) is the annoying one, as it concerns the total space and not the base. On the other hand, for applications to rationality problems, both (4) and (5) will disappear, see Theorem 7.10.

Definition 5.4. Let D be an Artin–Mumford component in the sense of Definition 5.1. We define an half-fiber above D as a $\mathbb{P}^1$ inside X defined in the following way. If D satisfies the hypothesis (1) of Definition 5.1 we take the fiber with its reduced structure X_p^{red}. If D satisfies the hypothesis (2) we take any of the two irreducible components of the cross above any closed point in D.

Remark 5.5. The name half fiber comes from the fact that, their class in cohomology is indeed half of the class of the fiber. This definition a priori does depend on the choice of the point p and not only on D, so it is a little abuse to call this *the* half fiber.

Definition 5.6. Keep notation from Theorem 5.2. Let D_1 and D_2 be the two connected components of the discriminant from hypothesis (2) and let ℓ_1 and ℓ_2 the associated half fibers as constructed in Definition 5.4. Let $\mathrm{cl} : \mathrm{CH}^{d-1}(X) \to \mathrm{H}^{2d-2}_{\mathrm{crys}}(X)$ be the cycle class map to crystalline cohomology. Define the class $\alpha \in \mathrm{H}^{2d-2}_{\mathrm{crys}}(X)$ as

$$\alpha := \mathrm{cl}(\ell_1) - \mathrm{cl}(\ell_2).$$

Lemma 5.7. *The class α is 2-torsion.*

Proof. We claim that $2\mathrm{cl}(\ell_i)$ is the class of a fiber. This will conclude the prove as 2α will then be the difference of two fibers, hence zero.

The claim is clear for a double line. In the case of a cross of lines, let us prove that the assumption on the nontriviality of $X_D \to D$ implies that $\mathrm{cl}(\ell_i) = \mathrm{cl}(m_i)$ where m_i is the other $\mathbb{P}^1$ in the same fiber. (This will imply that $2\mathrm{cl}(\ell_i) = \mathrm{cl}(\ell_i) + \mathrm{cl}(m_i)$ is indeed the class of a fiber.)

Let $\pi : \tilde{D}_i \to D_i$ the double cover trivializing the conic bundle on D_i. Let $\tilde{X}$ the normalisation of the pull-back of the conic bundle on $\tilde{D}_i$. Let us fix one $\mathbb{P}^1$ in $\tilde{X}$ which is sent isomorphically to ℓ_i and let us call it $\tilde{\ell}_i$. In the same fiber as $\tilde{\ell}_i$ the other irreducible component is denoted by $\tilde{m}_i$ and it is sent isomorphically to m_i. Let g be the involution on $\tilde{X}$ above X. Now, because the fibration over D_i is not trivial, we have that $g_*\mathrm{cl}(\tilde{\ell}_i) = \mathrm{cl}(\tilde{m}_i)$. If we push forward this relation we get

$$\mathrm{cl}(\ell_i) = \pi_*\mathrm{cl}(\tilde{\ell}_i) = (\pi \circ g)_*\mathrm{cl}(\tilde{\ell}_i) = \pi_*\mathrm{cl}(\tilde{m}_i) = \mathrm{cl}(m_i).$$

□

Lemma 5.8. *The class α from Definition 5.6 is nonzero.*

Proof. It is enough to show that the image of the class α via the natural application $\mathrm{H}^{2d-2}_{\mathrm{crys}}(X) \to \mathrm{H}^{2d-2}_{\mathrm{dR}}(X)$ is nonzero. To prove that α is nonzero in de Rham cohomology, it is enough to construct a class $\beta \in \mathrm{H}^2_{\mathrm{dR}}(X)$ such that $(\alpha, \beta) = 1$, where $(-,-) : \mathrm{H}^{2d-2}_{\mathrm{dR}}(X) \times \mathrm{H}^2_{\mathrm{dR}}(X)$ is the Poincaré duality pairing. On the other hand, by Corollary 4.6, there is an identification $\mathrm{H}^2_{\mathrm{dR}}(X) = \mathrm{H}^1(X, \Omega^1_{X/k})$, hence it is enough to construct a class $\beta \in \mathrm{H}^1(X, \Omega^1_{X/k})$. Such a class β can be taken to be as one of the classes appearing in Lemma 5.9(3) and we indeed have $(\alpha, \beta) = 1$ by Lemma 5.10. □

Lemma 5.9. *Let $s : B \to X$ be a degree 2 multisection of the conic bundle f. Define the class $\beta' \in \mathrm{H}^1(X, \Omega^1_X)$ as $\beta' := \mathrm{cl}(s(B))$. Consider the edge map*

$$\mathrm{Edge} : \mathrm{H}^1(X, \Omega^1_{X/k}) \to \mathrm{H}^0(B, R^1 f_* \Omega^1_X)),$$

for the Leray spectral sequence for $f : X \to B$

$$E_2^{a,b} := \mathrm{H}^a(B, R^1 f_* \Omega^1_{X/k}) \to \mathrm{H}^{a+b}(X, \Omega^1_{X/k}).$$

Then the following holds.

(1) Let $U \subset B$ the open subset on which $f : X \to B$ is smooth. Then, the restriction of $\mathrm{Edge}(\beta')$ to $\mathrm{H}^0(U, R^1 f_* \Omega^1_{X_U/k})$ is zero.
(2) There exists a unique class $\widetilde{\beta} \in \mathrm{H}^0(X, R^1 f_{\Omega_{X/k}})$ such that its restriction to $\mathrm{H}^0(X - D_1, R^1 f_{\Omega_{X/k}})$ vanishes and its restriction to $\mathrm{H}^0(X - (\coprod_{i \neq 1} D_i), R^1 f_{\Omega_{X/k}})$ equals to $\mathrm{Edge}(\beta')$.
(3) $\mathrm{Edge} : \mathrm{H}^1(X, \Omega^1_{X/k}) \to \mathrm{H}^0(B, R^1 f_* \Omega^1_{X/k}))$ is surjective. In particular, there exists a class β in $\mathrm{H}^1(X, \Omega^1_{X/k})$ such that $\mathrm{Edge}(\beta) = \widetilde{\beta}$.

Proof. By Proposition 4.4 there is a canonical isomorphism $R^1 f_* \Omega^1_{X_U/k} \simeq R^1 f_* \Omega^1_{X_U/U}$. In particular, thanks to Theorem 4.3, the coherent sheaf $R^1 f_* \Omega^1_{X_U/U}$ is locally free so that it is enough to show that the restriction of $\mathrm{Edge}(\beta')$ to the fiber X_η over the generic point $\eta \in B$ is zero. But this identifies with the class of a point of the smooth conic X_η defined over a degree 2 extension. Hence it is divisible by 2, hence it is zero in $\mathrm{H}^1(X_\eta, \Omega^1_{X/k(\eta)})$, which proves (1). Point (2) follows from (1) and the assumption on the discriminant.

The low degree terms from the Leray spectral sequence show that the obstruction to the surjectivity of part (3) is $\mathrm{H}^2(B, f_* \Omega^1_{X/k})$, which by Proposition 4.4 is isomorphic to $\mathrm{H}^2(B, \Omega^1_{B/k})$, which vanishes by assumption. □

Lemma 5.10. *The Poincaré pairing (α, β) equals 1.*

Proof. Since $(\alpha, \beta) = (\mathrm{cl}(\ell_1), \beta) - (\mathrm{cl}(\ell_2), \beta)$ is it enough to show that $(\mathrm{cl}(\ell_1), \beta) = 1$ and $(\mathrm{cl}(\ell_2), \beta) = 0$. To do this recall that $(\mathrm{cl}(\ell_i), \beta) = \beta_{|\ell_i} \in \mathrm{H}^1(\ell_i, \Omega^1_{\ell_i/k}) = k$ and that $\beta_{|\ell_i} = \mathrm{Edge}(\beta)_{|\ell_i}$. Now, we have

$$\beta_{|\ell_1} = \mathrm{Edge}(\beta)_{|\ell_1} = \mathrm{Edge}(\beta')_{|\ell_1} = (\mathrm{cl}(s(B)), \ell_1) = 1$$

since the multisection intersects in 1 point with multiplicity 1 the irreducible components of $f^{-1}(p_1)^{red}$ (as it intersect with multiplicity 2 any fiber). On the other hand

$$\beta_{|\ell_2} = \mathrm{Edge}(\beta)_{|\ell_2} = 0$$

since, by construction, $\mathrm{Edge}(\beta)$ vanishes on $\mathrm{H}^0(B - D_1, R^1 f_* \Omega_{X/k})$. □

6. Separation of the discriminant divisor

In order to apply Theorem 5.2 one needs, among other things, a conic bundle with disconnected discriminant. The goal of this section is the construction of a birational transformation allowing to modify a conic bundle with several irreducible components into one with several connected components. We fix an algebraically closed field k and a smooth connected k-variety B.

Elementary transformations. We recall here generalities on elementary transformations, see [Mar82, Section 1] for details. Let $T \subset B$ be a Cartier divisor, $\mathcal{F}$ be a nonzero vector bundle on T and $g : \mathcal{E}_{|T} \to \mathcal{F}$ be a surjection. The kernel of g is a subvector bundle $\mathcal{Y} \subset \mathcal{E}_{|T}$. The elementary transformation of $\mathcal{E}$ along $\mathcal{Y}$ is then the rank 3 subvector bundle $El_{\mathcal{Y}}(\mathcal{E}) \subset \mathcal{E}$ on B given by

$$El_{\mathcal{Y}}(\mathcal{E}) := \mathrm{Ker}(\mathcal{E} \to \mathcal{E}_{|T} \to \mathcal{F}).$$

The inclusion $El_{\mathcal{Y}}(\mathcal{E}) \subset \mathcal{E}$ induces a birational map $El_{\mathcal{Y}}(\mathcal{E}) \dashrightarrow \mathcal{E}$, which can be explicitly described Zariski locally and it is an isomorphism outside T, see [Mar82, Lemma 1.5].

More precisely, assume that $B = \mathrm{Spec}$ is affine, $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(A[a, b, c])$ and that T is defined by $t = 0$ for some $t \in A$. If $\mathcal{Y}$ is defined by t, a, then $\mathbb{P}(El_{\mathcal{Y}}(\mathcal{E}))$ is defined by $\mathrm{Proj}(A[ta, b, c])$ and the birational map $\mathbb{P}(El_{\mathcal{Y}}(\mathcal{E})) \dashrightarrow \mathbb{P}(\mathcal{E})$ is induced by sending a to ta. If $\mathcal{Y}$ is defined by t, a, b, then $\mathbb{P}(El_Y(\mathcal{E}))$ is defined by $\mathrm{Proj}(A[ta, tb, c])$ and the birational map $\mathbb{P}(El_Y(\mathcal{E})) \dashrightarrow \mathbb{P}(\mathcal{E})$ is induced by sending a to ta and b to tb.

Proposition 6.1. *Let $f : X \to B$ be a generically smooth conic bundle. Let Δ be the discriminant divisor (Definition 3.4). Assume the following.*

(1) One can write $\Delta = D_1 \cup D_2$ as the union of two closed subvarieties.
(2) D_1 and D_2 are smooth around $D_1 \cap D_2$ and intersect transversally.
(3) All the fibers above $D_1 \cap D_2$ are crosses.

Let P be the blow-up of B along $D_1 \cap D_2$ and E be the exceptional divisor. Then there exists a conic bundle $g : Y \to P$ whose discriminant divisor is the (disjoint) union of the strict transforms of D_1 and D_2 and such that the restrictions to $P - E$ of g and of the pull-back $f_P : X_P \to P$ of f coincide.

Proof. Let $q : S_2(\mathcal{E}) \to \mathcal{L}$ be the quadratic form corresponding to $f_P : X_P \to P$. By hypothesis the fibers of the restriction $f_E : X_E \to E$ are crosses. By [Tan24, Proposition 2.14(2)], the set $Y \subset \mathcal{P}(\mathcal{E}_{|E})$ of points living above E which are singular in the fiber is the projectivization of a rank 1 subvector bundle of $\mathcal{F}^\vee \subset \mathcal{E}^\vee_{|E}$. Consider the surjection $\mathcal{E}_{|E} \to \mathcal{F}$ and let $\mathcal{Y}$ be the kernel.

Let $\tilde{\mathcal{E}} := El_{\mathcal{Y}}(\mathcal{E}) \subset \mathcal{E}$ be the elementary transformation of $\mathcal{E}$ along $\mathcal{Y}$ and define $\tilde{q} : S_2(\tilde{\mathcal{E}}) \to \mathcal{L}$ to be the restrition of q to $S_2(\tilde{\mathcal{E}})$. Let $s \in \mathrm{H}^0(P, S^2(\tilde{\mathcal{E}}^\vee) \otimes \mathcal{L})$ be the section corresponding to $\tilde{q}$. We claim that its restriction $s_{|E}$ to E vanishes with order exactly 2. Assuming the claim s induce a section in $\mathrm{H}^0(B, S^2(\tilde{\mathcal{E}}^\vee) \otimes \mathcal{L} \otimes \mathcal{O}_P(-2E))$ which in return induces a quadratic form $\tilde{q}(E) : S_2(\tilde{\mathcal{E}} \otimes \mathcal{O}_P(E)) \to \mathcal{L}$. By construction, the new quadratic form $\tilde{q}(E)$ has discriminant divisor equal to the strict transform of Δ. Moreover, no modification has been done outside $P - E$, hence this will conclude the proof.

The claim can be proved locally in a neighborhood of E, so we can replace B with the completion of $\mathcal{O}_{B, D_1 \cap D_2}$ and then this with its strictly henselianisation A. By [Tan24, Proposition 2.14(2)], we can then assume that $X = \mathrm{Proj}(A[a, b, c]/\alpha c^2 - ba)$ for some $\alpha \in A$.

Let t_1, t_2 be the local parameter of D_1 and D_2. By assumption $\alpha = \alpha' t_1 t_2$, for some $\alpha' \in A^*$. Since $P \subset B \times \mathbb{P}^1_{\widetilde{t_1}, \widetilde{t_2}}$ is defined by $\widetilde{t_1} t_2 = \widetilde{t_2} t_1$, it will be enough to show that s vanishes with order exactly 2 on $t_1 = t_2 = 0$. By symmetry, it is enough to do the computation in the affine chart $\widetilde{t_1} = 1$.

In this chart, X_P has equations $\mathrm{Proj}(A[a,b,c]/\alpha'\widetilde{t}_2 t_1^2 c^2 - ba)$ and $\mathcal{Y}$ has equations $a = c = t_1 = 0$. Hence the elementary transformation $\tilde{\mathcal{E}}$ is

$$\mathrm{Proj}(A[a,b,c]/\alpha'\widetilde{t}_2 t_1^2 c^2 - t_1 b t_1 a),$$

which vanishes with order exactly 2 along $t_1 = 0$. □

Remark 6.2. We do not know a statement analogous to Proposition 6.1 under the assumption that the fibers on the intersection $D_1 \cap D_2$ might be nonreduced. A very general statement as Proposition 6.1 cannot be true, but it would be very useful to find conditions where such a birational separation exists. The main motivation is the construction of more examples to which Theorem 5.2 applies. For instance, it is very easy to construct conic bundles with reducible discriminant and such that all the singular fibers are nonreduced, see Remark 8.9.

7. An obstruction to the decomposition of the diagonal

In this section we recall the definition of the decomposition of the diagonal, which is intimately related to rationality question, following Voisin [Voi15] and [CTP16]. We give a cohomological obstruction to the decomposition of the diagonal (Proposition 7.6 and Proposition 7.9) and give geometric settings where this obstruction can be applied (Theorem 7.10, its corollary and Theorem 7.12).

Definition 7.1. Let X be a smooth proper geometrically connected scheme over a field. Consider the Chow ring $\mathrm{CH}(X \times X)$ of $X \times X$ with integral coefficients and the class of the diagonal $\Delta_X \in \mathrm{CH}(X \times X)$ in it. We say that X has decomposition of the diagonal if there exists a relation

$$\Delta_X = B_1 + B_2$$

in $\mathrm{CH}(X \times X)$ where the projection to the first factor of B_1 is supported on a scheme of dimension zero and the projection to the second factor of B_2 is not the whole X. If such a relation does not exist we say that X has no decomposition of the diagonal.

The relative version of the previous definition turns out to be the following.

Definition 7.2. A proper map $f : X \to Y$ over a field k is called universally CH_0-trivial if, for all field extension L/k, the pushfoward map induced by f on Chow groups $(f_L)_* : \mathrm{CH}_0(X_L) \to \mathrm{CH}_0(Y_L)$ is an isomorphism.

A proper variety Z over k is said to be universally CH_0-trivial if the structural map $f : Z \to \mathrm{Spec}(k)$ is so.

Theorem 7.3. [Voi15, CTP16].

(1) The group CH_0 *is a birational invariant for smooth proper varieties* [Ful98, Example 16.1.11]*, in particular being universally* CH_0*-trivial is a birational invariant for smooth proper varieties over a field.*

(2) For a smooth proper and geometrically connected variety over a field being universally CH_0*-trivial is equivalent to having the decomposition of the diagonal* [CTP16, Proposition 1.4]. *In particular having the decomposition of the diagonal is a birational invariant for smooth proper and geometrically connected varieties.*

(3) A variety which has no decomposition of the diagonal is not stably rational [CTP16, Lemma 1.5].

Definition 7.4. Let X and Y be proper varieties defined over a (possibly different) field. We say that X degenerates to Y if there exist a discrete valuation ring R and proper faithfully flat scheme over $\mathrm{Spec}(R)$ whose generic fiber is X and whose special fiber is Y.

Theorem 7.5. [CTP16, Theorem 1.14]. *Let X and Y be proper geometrically integral varieties with X smooth. Suppose tha X degenetes to Y and that Y admits a desingularization $\pi : \tilde{Y} \to Y$ such that π is universally CH_0-trivial. Suppose that Y has no decomposition of the diagonal (in the sense of Definition 7.1). Then X has no decomposition of the diagonal.*

Motivated by Theorem 5.2, we now explain how torsion in crystalline cohomology obstructs the decomposition of diagonal. This is the combination of [AV25, ABBvB21, CR11, Tot16].

Proposition 7.6. *Let k be an algebraically closed field of characteristic $p > 0$ and X be a smooth proper and connected k-variety. Assume that $\mathrm{H}^3_{\mathrm{crys}}(X)[p] \neq 0$. Then X has no decomposition of the diagonal.*

Proof. It is enough to combine [AV25, Theorem 1.1.5] with the following Theorem 7.7. □

Theorem 7.7. [ABBvB21, CR11, Tot16]. *Assume that X has decomposition of the diagonal. Then the following holds.*

(1) $\mathrm{H}^0(X, \Omega^i_X) = 0$ *for* $i \geq 1$.
(2) $Br(X) = 0$.
(3) $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ *for* $i \geq 1$.

Proof. Observe that (1) is [Tot16, Lemma 2.2] and (2) is [ABBvB21, Theorem 1.1]. For (3), let α be any class in $\mathrm{H}^i(X, \mathcal{O}_X)$. Consider the action of correspondence on Hodge cohomology and let us use the relation from Definition 7.1. We get

$$\alpha = \Delta_X^* \alpha = B_1^* \alpha + B_2^* \alpha.$$

Since the action of B_1 factors through a scheme of dimension zero, we get $B_1^* \alpha = 0$. Since B_2 does not dominate the second factor we have $B_2^* \alpha = 0$, by [CR11, Proposition 3.2.2]. This implies $\alpha = 0$ and proves (3). □

Lemma 7.8. *Let X be a smooth proper and connected variety of dimension d over an algebraically closed field k of characteristic $p > 0$. Then the k-vector spaces $\mathrm{H}^{2d-2}_{\mathrm{crys}}(X)[p]$ and $\mathrm{H}^3_{\mathrm{crys}}(X)[p]$ have the same dimension.*

Proof. This follows from Poincaré duality in the form stated in [Ber74, Théoréme 2.1.3, Page 555] and the universal coefficients theorem. □

Proposition 7.9. *Let X be a smooth proper and connected variety of dimension d over an algebraically closed field k of characteristic $p > 0$. Assume that $\mathrm{H}^{2d-2}_{\mathrm{crys}}(X)[p] \neq 0$. Then X has no decomposition of the diagonal.*

Proof. This is the combination of Proposition 7.6 and Lemma 7.8. □

Now Theorem 5.2 can be stated in a more geometric way.

Theorem 7.10. *Let k be an algebraically closed field of characteristic 2 and consider a flat conic bundle $f : V \to B$ over k with smooth generic fiber between smooth proper k-varieties. Assume the follwing.*

(1) The discriminant is disconnected.
(2) There are two disctinct connected components of the discriminant of f which are Artin–Mumford components (Definition 5.1).
(3) The group $\mathrm{H}^2(B, \Omega^1_{B/k})$ vanishes.

Then V has no decomposition of the diagonal.

Proof. This is the combination of Theorem 5.2 with Proposition 7.9 up to the fact that hypothesis (4) and (5) of Theorem 5.2 are no longer necessary. Indeed, if (5) is not satisfied then V has no decomposition of the diagonal (automatically, without the need of the other hypothesis) because of Theorem 7.7(1). Similarly, if (4) is not satisfied, then V has no decomposition of the diagonal by Theorem 7.7(1) combined with Corollary 4.5(2) and Theorem 7.7(3) combined with Corollary 4.5(1). □

Corollary 7.11. *Let V be as in Theorem 7.10. Let X be a smooth proper geometrically connected variety (defined over a field in characteristic zero or two). Suppose that X degenerates to a variety Y which is birational to V and which has a disingularization $\pi : \tilde{Y} \to Y$ such that π is universally CH_0-trivial (Definition 7.2). Then X has no decomposition of the diagonal.*

Proof. Having decomposition of the diagonal is a birational invariant (Theorem 7.3) so $\tilde{Y}$ has no decomposition of the diagonal by Theorem 7.10. We can conclude using Theorem 7.5. □

For conic bundles over surfaces the irrationality criterion can be made easier to apply (see Remark 7.13 for comments).

Theorem 7.12. *Let k be an algebraically closed field of charecteristic two, S be a smooth proper surface and $f : V \to S$ be a flat conic bundle with smooth generic fiber. Assume the following.*

(1) The group $\mathrm{H}^2(S, \Omega^1_{S/k}) = 0$.
(2) The discriminant divisor is reducible and the singular locus of each irreducible component of the discriminant is contained in the set of points whose fibers are double lines.
(3) The irreducible components of the discriminant meet transversally and the fibers above the intersections are crosses.
(4) There are at least two irreducible components which are Artin–Mumford in the sense of Definition 5.1.
(5) V is smooth around the fibers of double lines.

Then V has desingularization $\pi : \tilde{V} \to V$ such that π is universally CH_0-trivial and $\tilde{V}$ has no decomposition of the diagonal. In particular, any smooth variety (in characteristic zero or two) that degenerates to V has no decomposition of the diagonal hence it is not stably rational.

Proof. Consider the points $\{P_i\}$ in S of intersection of two components of the discriminant. Above each point P_i there is a unique point Q_i which is singular in the fiber (as the fiber is a cross). We claim that the singular points of V are exactly the $\{Q_i\}$ and that their are ordinary quadratic singularities. Under this claim we can resolve the singularities by simply blowing-up those points. Moreover the exceptional divisors are regular quadrics, hence rational, which implies that the map of desingularization is CH_0-trivial by [CTP16, Proposition 1.8]. On the other hand the resolution $\tilde{V}$ has no decomposition of the diagonal. Indeed by Proposition 6.1 it is birational to a conic bundle satisying the hypothesis of Theorem 7.10. We can conclude using Theorem 7.5.

We have to show the claim. This is essentially [Tan24, Theorem 2.14]. Indeed let us write t_1, t_2 for the formal coordinates around a fixed point P_i where the vanishing of a t_i corresponds to an irreducible component of the discriminant. Let a, b, c be the coordinates of the conic bundle. Then the proof in loc. cit. shows that the local equation of V in the tube above the formal neighborhood around P_i is $\alpha a^2 + bc = 0$ where α is the discriminant. Hence in this case we have the equation $t_1 t_2 a^2 + bc = 0$. This implies that the local affine equation around Q_i is $t_1 t_2 + bc = 0$, which gives the claim. □

Remark 7.13. One of the good features of Theorem 7.12 with respect to the previous ones, is that the discriminant does not need to be disconnected. Also the smoothness assumption on the total space is easier to check as it is reduced to the hypothesis (5) (and partially (2)).

8. Concrete examples

In this section we construct explicit examples where Theorem 7.12 applies and deduce from it irrationality results, both in characteristic two and zero (Theorem 8.5). All these examples are in characteristic two and have $\mathbb{P}^2$ as base. We will use x, y, z for the coordinates on the base $\mathbb{P}^2$. The points on which the fibers are double lines will always be $[0:1:0]$ and $[0:0:1]$. The total space will be regular above these two points. The discriminant will have two components, each one passing in exactly one of these two points and being singular only there.

Once the examples are found, verifying that they do satisfy these conditions (and hence the hypothesis of Theorem 7.12) is an elementary computation based on the Jacobian criterion and on the description of the loci Δ and Σ (of Definition 3.4) using formulas from Remark 3.6. We do not write these computations.

The way the examples were found was by first guessing the possible discriminants and then finding the coefficients that such a conic bundle should have. With no surprise then, the former have easier equations than the latter. To guess discriminants, it was useful to have the restrictions from [ABBGvB21, Theorem 4.3] describing the local behaviour of Δ around Σ.

We also write a last example, taken from [ABBvB21], where one of the two components of the discriminant has only crosses but the family over it is not a product. That example was found using Magma, as the authors explain. The coefficients there are easier than the discriminant.

Example 8.1. Let $B = \mathbb{P}^2$ with coordinates x, y, z. Consider the conic bundle defined by

	$\mathcal{O}$	$\mathcal{O}(1)$	$\mathcal{O}(3)$
$\mathcal{O}$	1		
$\mathcal{O}(1)$	x	zy	
$\mathcal{O}(3)$	0	$x(y^3+z^3)+y^2z^2$	$y^6+z^6+x^4yz+xz^5+xy^5$

then the discriminant is

$$\Delta = (x^3z + y^4)(x^3y + z^4).$$

Remark 8.2. Let us explain how to find such an example. Write

	$\mathcal{O}$	$\mathcal{O}(1)$	$\mathcal{O}(3)$
$\mathcal{O}$	1		
$\mathcal{O}(1)$	x	α	
$\mathcal{O}(3)$	0	β	γ

for a general matrix of this form. The discriminant is $x^2\gamma + \beta^2$. First notice that α does not appear in the discriminant: its choice will only matter for the smoothness of the total space. Then one is looking for the following two conditions.

(1) $\beta = x = 0$ consists exactly of the two points $[0 : 1 : 0]$ and $[0 : 0 : 1]$.
(2) $(x^3z + y^4)(x^3y + z^4) + \beta^2$ is divisible by x^2: the quotient will then be γ.

Relation (2) simply becomes: β is y^2z^2 modulo x.

Example 8.3. Let $B = \mathbb{P}^2$ with coordinates x, y, z and let g be an homogenous polynomial of degree 2.

	$\mathcal{O}$	$\mathcal{O}(1)$	$\mathcal{O}(3)$
$\mathcal{O}$	1		
$\mathcal{O}(1)$	x	g	
$\mathcal{O}(3)$	0	$x(y^3 + z^3) + y^2z^2$	$y^6 + z^6 + x^4yz + xz^5 + xy^5$.

Since the term g does not appear in the equation of the discriminant the factorization

$$\Delta = (x^3z + y^4)(x^3y + z^4)$$

still holds. Hence, every time this conic bundle is smooth around $[0 : 1 : 0]$ and $[0 : 0 : 1]$ one can apply Theorem 7.10. Observe that, the open set of polynomials g satisfying this smoothness condition is nonempty by the previous example.

Example 8.4. $B = \mathbb{P}^2$ with coordinates x, y, z. Consider the conic bundle with value in $\mathcal{O}(1)$ defined by

	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(3)$
$\mathcal{O}(1)$	$x + y$		
$\mathcal{O}(1)$	x	x	
$\mathcal{O}(3)$	0	$x(y^2 + z^2) + y((x + z)z + (z + y)y)$	f

where

$$f = x^2yz^2 + x^2z^3 + xy^4 + xy^3z + xz^4 + y^5 + y^2z^3 + yz^4 + z^5.$$

Then the discriminant is

$$\Delta = (x^2z + xy^2 + y^3)(x^2yz + x^2z^2 + y^4 + y^2z^2 + z^4).$$

This example can be found following the strategy from Remark 8.2.

Theorem 8.5. *Let k be a field of characteristic zero or two. A very general conic bundle in $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(3))$ over $\mathbb{P}^2_k$ with values in $\mathcal{O}$ has no decomposition of the diagonal, hence it is not stably rational.*

Similarly, a very general conic bundle in $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(3))$ over $\mathbb{P}^2_k$ with values in $\mathcal{O}(1)$ has no decomposition of the diagonal, hence it is not stably rational.

By very general we mean that the coefficients of the polynomials defining the conic bundle are algebraically independent over the prime field.

Proof. It is enough to degenerate such a very general conic bundle to the above examples 8.1 and 8.4 and apply Theorem 7.12. □

Example 8.6. $B = \mathbb{P}^2$ with coordinates x, y, z. Consider the conic bundle defined by

	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
$\mathcal{O}(1)$	$y^2 + j^2z^2$		
$\mathcal{O}(1)$	xz	yz	
$\mathcal{O}(1)$	$jx^2 + yz$	xy	$z^2 + jy^2$

with j such that $j^2 + j + 1 = 0$. Then the discriminant is

$$\Delta = (x^2z + y^3)(x^2y + z^3)$$

Remark 8.7. In order to find such an example, the strategy from Remark 8.2 has to be modified because there is no zero in the matrix. The idea is similar, write

	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
$\mathcal{O}(1)$	α		
$\mathcal{O}(1)$	γ'	β	
$\mathcal{O}(1)$	β'	α'	γ

for a general matrix of this form. First impose that the conics $\alpha', \beta'$and γ' meet exactly in $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Then complete the matrix so that the discriminant has the desired form $\Delta = (x^2z + y^3)(x^2y + z^3)$. (As already mentioned, such form is guessed based on [ABBGvB21, Theorem 4.3].) It is harder to implement this strategy.

An example of this shape might exists over $\mathbb{P}^3$ (i.e. the quadric surfaces $\alpha', \beta'$and γ' meet in a finite number of points and the discriminant has two irreducible components above which only crosses lie.) This would be of great interest as one would find a variety of dimension four which is irrational because of an H^3. Arguing through weak Lefschetz one could hope to have examples in any dimension.

The above example also gives the stable irrationality of some very general conic bundles but this is already in [ABBvB21], based on the following example (and a different rationality obstruction).

Example 8.8. [ABBvB21, Section 6] Consider the conic bundle

	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
$\mathcal{O}(1)$	$xy + xz + z^2$		
$\mathcal{O}(1)$	$x^2 + xz + z^2$	$x^2 + yz + z^2$	
$\mathcal{O}(1)$	xy	$x^2 + yz + z^2$	$y^2 + xz + z^2$

with discriminant

$$\Delta := xz(x + z)(y^2x + x^2y + xyz + z^2x + y^3).$$

The fibration is not trivial on $(y^2x + x^2y + xyz + z^2x + y^3)$ by [ABBvB21, Lemma 6.8 and Proposition 6.9]. The other three component of the discriminant meet in exactly one point whose fiber is a double line (and the total space is regular there).

Remark 8.9. In characteristic two, it is very easy to construct examples of conic bundles over $\mathbb{P}^n$ (for any n) with reducible discriminant and only double lines over it. It is enough to take a conic bundle of the form

$$\begin{array}{c|ccc} & \mathcal{O} & \mathcal{O}(n_1) & \mathcal{O}(n_2) \\ \hline \mathcal{O} & 1 & & \\ \mathcal{O}(n_1) & ab & c & \\ \mathcal{O}(n_2) & 0 & 0 & d \end{array}$$

and check that the discriminant (with reduced structure) is ab and has the desired properties. For generic choices of c, d the total space of the conic bundle will also be regular (except above the intersection of the two components $a = 0$ and $b = 0$). Unfortunately we do not know how to link any of such a construction to Theorem 7.10. What is missing is a birational transformation allowing to separate the two components $a = 0$ and $b = 0$, see Remark 6.2.

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